Exercise 2.5.5

(A general example of non-uniqueness) Consider the initial value problem $\dot{x} = |x|^{p/q}$, $x(0) = 0$, where p and q are positive integers with no common factors.

a) Show that there are an infinite number of solutions for $x(t)$ if $p < q$.

b) Show that there is a unique solution if $p > q$.

Solution

Part (a)

Suppose that $p < q$. Then

$$q - p > 0 \quad \rightarrow \quad \frac{q-p}{q} > \frac{0}{q} \quad \rightarrow \quad 1 - \frac{p}{q} > 0.$$

Split up the ODE over the intervals where the absolute value function is defined and separate variables to find $x(t)$.

$$\begin{aligned} \frac{dx}{dt} &= (-x)^{p/q}, \quad x < 0 & \frac{dx}{dt} &= x^{p/q}, \quad x \geq 0 \\ \frac{dx}{(-x)^{p/q}} &= dt & \frac{dx}{x^{p/q}} &= dt \\ (-x)^{-p/q}\,dx &= dt & x^{-p/q}\,dx &= dt \\ \int (-x)^{-p/q}\,dx &= \int dt & \int x^{-p/q}\,dx &= \int dt \\ \frac{1}{1-\frac{p}{q}}(-x)^{1-p/q} &= t + C_1 & \frac{1}{1-\frac{p}{q}}x^{1-p/q} &= t + C_2 \end{aligned}$$

Combine the two equations.

$$\frac{1}{1-\frac{p}{q}}|x|^{1-p/q} = t + C_3 \tag{1}$$

Apply the initial condition $x(0) = 0$ to determine the integration constant.

$$\frac{1}{1-\frac{p}{q}}(0)^{1-p/q} = (0) + C_3 \quad \rightarrow \quad C_3 = 0$$

As a result, equation (1) becomes

$$\frac{1}{1-\frac{p}{q}}|x|^{1-p/q} = t.$$

Solve for $x(t)$.

$$|x|^{(q-p)/q} = \left(1 - \frac{p}{q}\right)t$$

$$|x| = \left[\left(1 - \frac{p}{q}\right)t\right]^{q/(q-p)}$$

Therefore,

$$x(t) = \pm \left[\left(1 - \frac{p}{q} \right) t \right]^{q/(q-p)}.$$

Notice that $x(t) = 0$ is also a solution of the ODE and its associated initial condition. Construct another solution depending on the parameter t_0.

$$x(t) = \begin{cases} 0 & \text{if } t < t_0 \\ \pm \left[\left(1 - \frac{p}{q} \right) (t - t_0) \right]^{q/(q-p)} & \text{if } t > t_0 \end{cases}$$

Since t_0 is arbitrary, there are an infinite number of solutions to the initial value problem.

Part (b)

Suppose that $p > q$. Then

$$p - q > 0 \quad \rightarrow \quad \frac{p-q}{q} > \frac{0}{q} \quad \rightarrow \quad \frac{p}{q} - 1 > 0,$$

and the solution to the ODE obtained by separating variables becomes

$$\begin{aligned} |x| &= \left[\left(1 - \frac{p}{q} \right) t \right]^{q/(q-p)} \\ &= \left[- \left(\frac{p}{q} - 1 \right) t \right]^{-q/(p-q)} \\ &= (-1)^{-q/(p-q)} \left[\left(\frac{p}{q} - 1 \right) t \right]^{-q/(p-q)}. \end{aligned}$$

The right side is potentially negative; for example, choosing $q = 1$ and $p = 2$ results in

$$|x| = (-1)^{-1} t^{-1} = -\frac{1}{t}.$$

Consequently, this solution must be discarded. Only $x(t) = 0$ satisfies the initial value problem in the case that $p > q$, meaning it's a unique solution.